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The Geocentric Orientation Vector for the Australian Geodetic Datum

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Summary

The geocentric orientation vector is defined for the Australian Geodetic Datum on the assumption that the geoid has the same potential as the value on Reference Ellipsoid 1967 on adopting Reference System 1967. It is shown that only the free air geoid need be considered to provide the required definition in the Australian region with a precision equivalent to that of the data set currently available. Solutions obtained by two different techniques indicate that the parameters required to define the vector at the Johnston origin of the datum are

$$\Delta\xi_o = -4{\cdot}2 \pm 0{\cdot}2\,\mathrm{s}$$
$$\Delta\eta_o = -4{\cdot}5 \pm 0{\cdot}2\,\mathrm{s}$$
$$\Delta N_o = 7{\cdot}2 \pm 0{\cdot}2\,\mathrm{m}.$$

The error estimate in the last parameter also assumes that no significant errors exist in low degree harmonics of degree $n(< 5)$ and orders zero and one used in the representation of the Earth's gravity field. Such errors cannot be detected over the 2 per cent of the Earth's surface area included in the present study. The consequent errors are unlikely to exceed ± 3 m on current estimates of the accuracy of low degree harmonic coefficients.

A guide to notation

Commonly used symbols

a = Equatorial radius of reference spheroid

A = Azimuth

f = Flattening of the meridian ellipse

h = Elevation

h_i = Linearization parameter.

h_n = Normal elevation.

h_o = Orthometric elevation.

k = Gravitational constant.

$M\{x\}$ = Mean value of x.

N = Geoid/spheroid separation.

r = Distance between variable element dS and computation point P.

R = Mean radius of the Earth.

U = Potential of the reference system.

V_d = Disturbing potential (Potential anomaly).

W = Potential of the existent earth (Geopotential).

x_i $(i = 1, 3)$ = A general rectangular Cartesian co-ordinate system in Earth sp with a local origin.

α = A parameter associated with azimuth.

β = Ground slope.

γ = Mean value of normal gravity over the reference spheroid.

Δx = A small change in x.

Δg = Gravity anomaly, which, to the order of the flattening, is the air anomaly.

$\Delta\xi_i$ $(i = 1, 3)$ = Set of curvilinear geocentric orientation parameters which de the orientation vector in relation to a local Cartesian system the general point on the geodetic datum. At the origin,

$$\Delta\xi_{1_o} = \Delta\xi_o; \quad \Delta\xi_{2_o} = \Delta\eta_o; \quad \Delta\xi_{3_o} = \Delta h_{so}.$$

η = Component of deflection of the vertical in the prime verti positive if the outward vertical is east of the normal; represe by ξ_2 in text.

λ = Longitude, positive east.

ξ = Component of deflection of the vertical in the meridian, positi outward vertical is north of spheroid normal; represented a in text.

ξ_i $(i = 1, 3)$ = Set of curvilinear parameters defining the separation vector.

σ_i $(i = 1, 3)$ = Root mean square (rms) residual of comparisons.

ϕ = Latitude, positive north.

ψ = Angular distance on unit sphere between the variable elen dS and P.

Subscripts

Subscripts which are not indices are introduced with the intention of kee the number of variables down to a minimum and simplifying the written form of equations and hence improving the comprehension of concepts.

a = Astronomically determined values; astro-geodetic values.

d = Disturbing value; the difference between equivalent values on the true reference systems.

c = Corrections to the free air geoid.

f = Free air geoid values.

g = Geocentric values; gravimetric values.

o = Values at the origin of the geodetic datum.

p = Evaluated at the fixed point P.

q = Values at the variable element ds.

s = Referred to the spheroid.

Miscellaneous points

AGD = Australian Geodetic Datum.

RS 1967 ≐ Reference System 1967.

UNSW set = The anomaly set prepared from the gravity holdings for the Australian region at the University of New South Wales.

i = Unit vector along the x_i axis.

N = Unit normal vector.

O = Geocentric orientation vector defined by the equations

$$\mathbf{O} = \sum_{i=1}^{3} h_{io}\, \Delta\xi_{io}\, \mathbf{i}_o = \sum_{i=1}^{3} h_i\, \Delta\xi_i\, \mathbf{i}. \qquad (1)$$

1. Introduction

The Australian Geodetic Datum (AGD) has been established by the Commonwealth of Australia's Division of National Mapping with the prime purpose of providing a first-order geodetic framework for topographic mapping programmes. It is defined by the following parameters:

(a) The Australian National Spheroid (ANS) given by

$$\left.\begin{aligned} a &= 6\,378\,160\,\mathrm{m} \\ f^{-1} &= 298{\cdot}25 \end{aligned}\right\} \qquad (2)$$

and

(b) the geodetic co-ordinates and the spheroidal elevation adopted at the Johnston origin of the datum, being (Lambert 1968, p. 95)

$$\begin{aligned} \phi_o &= -25°\ 56'\ 54{\cdot}5515'' \\ \lambda_o &= 133°\ 12'\ 30{\cdot}0771'' \\ h_{so} &= 571{\cdot}2\,\mathrm{m}. \end{aligned}$$

A review of the progressive stages in the definition of this datum is available (Mather & Fryer 1970b), a study of which shows that the geodetic co-ordinates at the origin are based on the mean values of the deflections of the vertical at approximately 150 astro-geodetic stations available in 1963 and well spaced over the six and one half million square kilometre continental area. The adoption of these numerical means as a datum correction has the effect of fitting the reference spheroid to an *estimate* of the mean geoid slope as defined by the directions of the vertical at the astro-geodetic stations included in the analysis. Also see Section 4(i). This procedure is a not uncommon geodetic practice when working over limited areas as it has the considerable advantage of enabling reduction of measurements to the *geoid* to be considered appropriate for the purpose of computations on the spheroid. This technique, called the *development method* (Molodenskii, Eremeev & Yurkina 1962, p. 29), ignores the consequences of the geoid separating from the spheroid. The condition of parallelism between the surfaces ensures negligible errors in the geodetic co-ordinates computed on such a system.

A geodetic control network on the AGD can therefore be assumed to meet first-order requirements as the system is controlled by over 1000 Laplace stations which have a continent-wide distribution. A preliminary astro-geodetic geoid was prepared by Fischer & Slutsky (1967) based on 600 of these stations, the locations of which are shown in (Bomford 1967, p. 56). This solution is based on an average station density of one station per 10 000 km^2 and understandably produces an over-smoothened representation of geoid undulations on the AGD, being studied in greater detail in Section 4(ii). It does however show the major features of the geoid with considerable accuracy and provides a useful independent check on the accuracy of gravimetric solutions of the geoid. These comparisons, in turn, can be used for computing the corrections necessary for the definition of the AGD in relation to the Earth's centre of mass (geocentre). Such a definition is essential in the context of global geodesy which has, as one of its prime goals, the determination of the physical surface of the Earth in terms of the other invariants defining earth space. The Earth's gravitational field and its associated inertia tensors (e.g., Hotine 1969, p. 164 *et seq*) can be considered to comprise these invariants to the order of accuracy sought in the present study.

The problem can be visualized as one in which the centre of the local astro-geodetic spheroid was at some point C' other than the geocentre C as illustrated in Fig. 2. The displacement $C'C$ in earth space can be uniquely represented by a vector and called the *geocentric orientation vector* (O) in the present study. This vector can either be recovered as a by-product of a global space triangulation scheme or by the use of the Earth's gravity field. A gravimetric solution for the geoid in Australia was performed in 1968 (Mather 1969, p. 499 *et seq*.). Comparisons with the solutions of Fischer & Slutsky showed the existence of significant systematic errors in certain regions which, from a study of their surface disposition, were attributed to the incompatibility between the four independent data sets used in the representation of the global gravity field. The exclusion of these regions reduced the root mean square (rms) residual (σ_3) of the comparisons after translation of the reference frame of the astro-geodetic solution and given by

$$\sigma_i^2 = M\{(\xi_{ia}-\xi_{ig})^2\}, \qquad i = 1, 3 \tag{3}$$

from $\pm 5{\cdot}2$ m to $\pm 3{\cdot}0$ m (*ibid*, p. 513). (Note: For significance of undefined variables and subscripts, see 'Guide to notation'.)

The gravimetric solution used in this study was the free air geoid which has been shown to be a good approximation to the geoid provided there are no zero degree linear contributions to the indirect effect (Mather 1968a, p. 523). The 1968 solution emphasized the critical significance of internal consistency in the gravity data sets which therefore had to be made compatible prior to any determination of the geocentric orientation vector. This effect is a consequence of attempting to represent a continuous field from an irregularly spaced set of discrete gravity measurements, the resulting prediction errors having systematic components of practical consequence. It is, therefore, prudent not to restrict the solution for the parameters defining the orientation vector for the datum under investigation to a single point on it, even though such a determination is possible in theory.

The solution, in principle, is as follows. The geocentric orientation parameters $\Delta\xi_i$ and their associated linearization parameters h_i ($i = 1, 3$) define the magnitude O of the orientation vector by the equation

$$O = \sum_{i=1}^{3} h_i^2\, \Delta\xi_i^2 = \sum_{i=1}^{3} h_{io}^2\, \Delta\xi_{io}^2, \tag{4}$$

where

$$\Delta\xi_i = \xi_{ig}-\xi_{ia}, \qquad i = 1, 3. \tag{5}$$

ξ_i ($i = 1, 3$) in the above equations comprise a set of parameters which can be defined on both the local geodetic datum (ξ_{ia}) as well as the geocentric one (ξ_{ig}) and are a function of the reference frame adopted which should have the same dimensions in both cases but with different locations in earth space. The latter quantities can be defined by either the use of geometrical satellite methods over global extents or gravimetric techniques. The orientation vector can be expressed either by equation (4) or by the changes Δu_i in earth space co-ordinates given by

$$\Delta u_i = u_{ig} - u_{ia} = \sum_{j=1}^{3} a_{ij}(\xi_{jgo} - \xi_{jao}) \qquad i = 1, 3, \tag{6}$$

where the terms a_{ij} define the rotational parameters. Equation (6) can be used either for the evaluation of the differences $\Delta\xi_{io}$ $(i = 1, 3)$ at the origin or for the computation of Δu_i at all stations on the datum if the former were known. Equation (6) cannot be used directly as the results will be prone to the effect of systematic errors in the representation adopted for the gravity field. The problem is complicated further by the fact that continental areas have well-defined gravity fields while ocean areas are inadequately surveyed for geodetic purposes. In addition, local areas have larger effects on computations than more distant regions.

It is therefore preferable to confine investigations to continental extents which are not too close to un-surveyed oceans. The scheme finally adopted was based on the following conclusions.

(i) Investigations should not be confined to either a single station or a restricted portion of the region covered by the datum.

(ii) It was therefore desirable to evaluate the parameters $\Delta\xi_{io}$ $(i = 1, 3)$ defining the orientation vector through equations (1) and (4) by the use of comparisons of the type at equation (5) in accordance with (i) provided the near zone gravity field was adequately defined.

The present investigation develops the relations necessary to define the quantities $\Delta\xi_{io}$. The specifications for the digital representation of the incompletely surveyed gravity field are given in Section 3. The results obtained are analysed for the best values of $\Delta\xi_{io}$ at the Johnston origin together with estimates of the accuracy attained.

2. Theoretical concepts

The solution of the problem defined in Section 1 is derived at length in (Mather 1970) as a refinement and development of the principles outlined in (Mather 1968a, p. 526 *et seq.*). Only a summary of formulae and conditions of solution will be given here, copies of the report being obtainable from the writer. The problem can be defined in the following three stages.

(i) *The gravimetric solution*

The height anomaly (h_d) between the physical surface and the telluroid (see Fig. 1) can be completely defined by the equation

$$h_{dp} = N_{fp} + N_{cp} + o\{f h_d\}, \tag{7}$$

where the free air geoid separation at P is given by

$$N_{fp} = \frac{W_o - U_o}{\gamma} - R\frac{M\{\Delta g\}}{\gamma} + \frac{1}{4\pi\gamma R}\int\int f(\psi)\,\Delta g\,dS, \tag{8}$$

$f(\psi)$ being Stokes' function (e.g., Heiskanen & Moritz 1967, p. 94), W_o and U_o being the potentials of the geoid and spheroid respectively, while $M\{\Delta g\}$ is the global mean value of the gravity anomaly. All other quantities are defined in the ' Guide

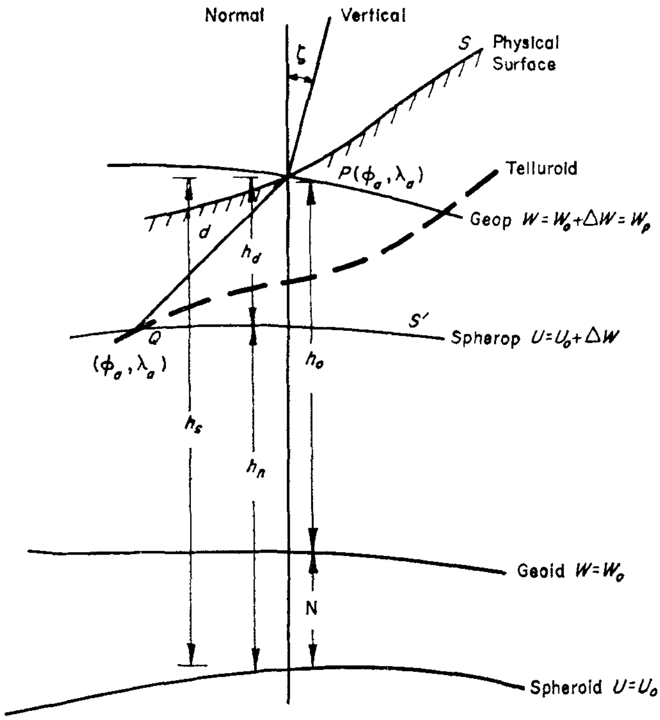

FIG. 1. The telluroid in relation to the physical surface. d = separation vector.

to notation'. N_{cp} in equation (7) is given by

$$N_{cp} = \frac{1}{2\pi\gamma} \int\int \left(\left(\frac{R\sin\psi}{r^3}\frac{dh}{dr} + \frac{h_p - h}{r^3}\right) V_d - \frac{\gamma}{r}\sum_{j=1}^{2} \xi_j \tan\beta_j\right) dS,$$

$$r < 200\,\mathrm{km}, \quad (9)$$

where

$$\frac{dh}{dt} = \sum_{j=1}^{2} \tan\beta_j \cos\alpha_{qj}; \quad (10)$$

$$\alpha_{q1} = A_q; \qquad \alpha_{q2} = \tfrac{1}{2}\pi - A_q. \quad (11)$$

h in equations (9) and (10) refer to orthometric elevations, the disturbing potential V_d being given by

$$V_d = W_o - U_o + \gamma h_d + o\{f V_d\}, \quad (12)$$

while the sign convention of the deflections of the vertical ξ_i ($i = 1, 2$) is defined in the 'Guide to notation'. β_i ($i = 1, 2$) are the gradients of the topography in the north and east directions at dS while A_q is the azimuth of P from dS.

The gravimetric solution so defined is correct to the order of the flattening and can be considered to consist of a spherical Stokesian term N_f which requires global evaluation, together with a correction term N_c whose effect, to this same order, is local. Its magnitude for Mt Blanc (*ibid.*, p. 329) as computed by Arnold is $-0{\cdot}2$ metres. The restriction of the range of r in equation (9) to within 200 km of the computation point cannot affect the result to more than the order of the flattening as the physical surface satisfies the boundary conditions of Stokes' problem to this order (Jeffreys 1962, p. 192). The solution expressed by equations (7)–(12) have a simple interpretation in terms of earth space invariants as will be seen in Section 2(ii).

The surface deflections of the vertical ξ_i at P are obtained with respect to the normal of the associated spherop at Q in Fig. 1 according to the equation

$$\xi_{ip} = \xi_{fip}+\xi_{cip}, \qquad i = 1, 2, \tag{13}$$

where the contribution of the free air geoid is given by

$$\xi_{fip} = \frac{1}{4\pi\gamma} \int\int \frac{\partial}{\partial\psi} \{f(\psi)\}\, \Delta g \cos\alpha_i \, dS, \qquad i = 1, 2, \tag{14}$$

$$\alpha_1 = A; \qquad \alpha_2 = \tfrac{1}{2}\pi - A, \tag{15}$$

A being the azimuth of dS from P. For most practical purposes, the correction term can be given by

$$\xi_{cip} = \frac{1}{2\pi\gamma R^3} \int\int \left(\left(\frac{R\gamma}{\psi^2}\sum_{j=1}^{2} \xi_j \tan\beta_j - \left\{2\frac{dh}{dr}+3\frac{h_p-h}{R\psi}\right\}\frac{V_d}{\psi^3}\right)\cos\alpha_i + (-1)^i \frac{\partial}{\partial A_c}\left(\frac{dh}{dr}\right)\frac{V_d}{\psi^3}\sin\alpha_i\right) dS, \qquad i = 1, 2; \qquad \psi < 2^\circ, \tag{16}$$

where

$$\frac{\partial}{\partial A_c}\left(\frac{dh}{dr}\right) = \sum_{j=1}^{2} (-1)^j \sin\alpha_{qj} \tan\beta_j. \tag{17}$$

The surface integrals at 8, 9, 14 and 16 are indeterminate at $\psi = 0$. The general principles for the evaluation of such inner zone effects are set out in (Mather 1970, pp. 41 *et seq.*). The solutions for equations (8) and (14) are well known (e.g. Heiskanen & Moritz 1967, p. 122) while those for equations (9) and (16) are

$$N_{cpin} = -r_o\left(\sum_{j=1}^{2} \xi_j \tan\beta_j\right)_{r=0}, \tag{18}$$

where r_o is the radius of the innermost zone, assumed circular, and

$$\xi_{cipin} = 0, \qquad i = 1, 2, \tag{19}$$

on assuming both the topography and the geops to be planar over the inner zone.

ξ_{cip}, unlike N_{cp}, can have significant magnitudes primarily in regions where the topography is undulating and has asymmetric variations with respect to the computation point. 'Abnormal' deflections of the vertical should therefore be expected in regions on the peripheries of mountain ranges which rise above great plains. As a corollary, no significant corrections should be necessary to the free air geoid terms in high plateaus. Deflections of the vertical with reference to a geocentric reference spheroid (ξ_{gi}, $i = 1, 2$) are obtained by allowing for the curvature of the spherop normal when

$$\xi_{gi} = \xi_i + c_{\xi i} = \xi_{fi}+\xi_{ci}+c_{\xi i}, \qquad i = 1, 2, \tag{20}$$

where

$$c_{\xi 1} = 0{\cdot}17\, \mathrm{h}^{(km)} \sin 2\phi \sec; \qquad c_{\xi 2} = 0, \tag{21}$$

ϕ, h being the latitude and elevation of the point in question.

(ii) *Interpretation of the gravimetric solution*

The significant contribution to the gravimetric solution is the free air geoid separation N_f which contains the Stokesian term, the surface integral being valid if the gravity anomaly at the physical surface has no first degree harmonic on global analysis, the separation so defined in this case being referred to a spheroid whose centre is at the geocentre. Conversely, the use of Stokes' integral with any other anomaly will define a separation with respect to a spheroid centred at the centre of mass of an amended mass distribution which gives rise to this type of anomaly as the gravity anomaly at the physical surface. The use of anomalies of the type suggested by Molodenskii and Moritz (1968) cannot deviate from the geocentric condition by more than the order of the flattening. The exclusion of the harmonics (2, 1) imposes the condition that the minor axis of the spheroid coincides with the principal axis of greatest moment of inertia of the Earth which, for all practical purposes, is the mean rotation axis. These conditions which are completely developed in (Mather 1970, pp 45–55), are well known (e.g. Heiskanen & Moritz 1967, p. 62) and it is conventional to make appropriate allowance when preparing global anomaly sets (e.g. Rapp 1969a, p. 78).

(iii) The comparison of gravimetric and astro-geodetic solutions

Two stages are involved in the comparison. In the first, the quantities (ξ_i, $i = 1, 3$) to be compared on each of the systems are made compatible by adopting a common spheroid of reference. In the current solution, the gravity data referred to the International Gravity Formula (e.g., Heiskanen & Moritz 1967, pp 79 & 80) was converted to RS 1967 (IAG Resolutions 1967, p. 367) by a set of differential formulae (Mather 1968b) as the spheroid implicit in the latter has the same dimensions as the Australian National Spheroid defined in equation (2). Alternately, the changes in ξ_i on revising the dimensions of the spheroid could be treated purely geometrically (e.g., Vening Meinesz 1950).

The second stage is concerned with the changes in earth space co-ordinates on a shift of the reference spheroid as outlined in Section 1. The gravimetric solution, as discussed in Section 2(ii) above, is referred to a geocentric spheroid which has its minor axis coincident with the Earth's rotation axis. The earth space location of the astro-geodetic datum is defined by the direction cosines of the surface verticals in the local region. It can easily be shown (e.g., Mather 1970, p. 63) that six parameters have to be 'arbitrarily' defined to complete the earth space location of the latter datum. This can be interpreted in a number of ways. The adoption of the definitions at equation (2), with the implicit equality of the minor axis direction cosines with those of the rotation axis, results in the geodetic spheroid having its centre at some point C' in earth space (Fig. 2) which is not coincident with the geocentre C.

The geocentric orientation vector $\mathbf{O} = \overrightarrow{C'C}$ can be completely represented by three components on any three-dimensional Cartesian co-ordinate system in earth space. The adoption of such a system $(x_1\, x_2\, x_3)$ in the local Laplacian trihedron, defined by the north (x_1) and east (x_2) directions in the local geodetic horizon and its outward normal (x_3) at the general point on the datum, affords the relations

$$\mathbf{O} = \overrightarrow{C'C} = \sum_{i=1}^{3} \Delta x_i \, \mathbf{i} = \sum_{i=1}^{3} h_i \, \Delta\xi_i \, \mathbf{i}, \tag{22}$$

where the triad of unit vectors $\mathbf{i}$ ($i = 1, 3$) define the earth space orientation of the local Laplacian system, the quantities h_i and $\Delta\xi_i$ having the same definition as in equations (4) and (5). The addition of the subscript $_o$ refers to evaluation at the origin of the geodetic datum. If ξ_1 and ξ_2 are defined as in equation (20) and ξ_3 is interpreted as h_d

$$\left.\begin{aligned} h_1 &= -(\rho + h_n + h_d) \\ h_2 &= -(\nu + h_n + h_d) \\ h_3 &= 1 \end{aligned}\right\}, \tag{23}$$

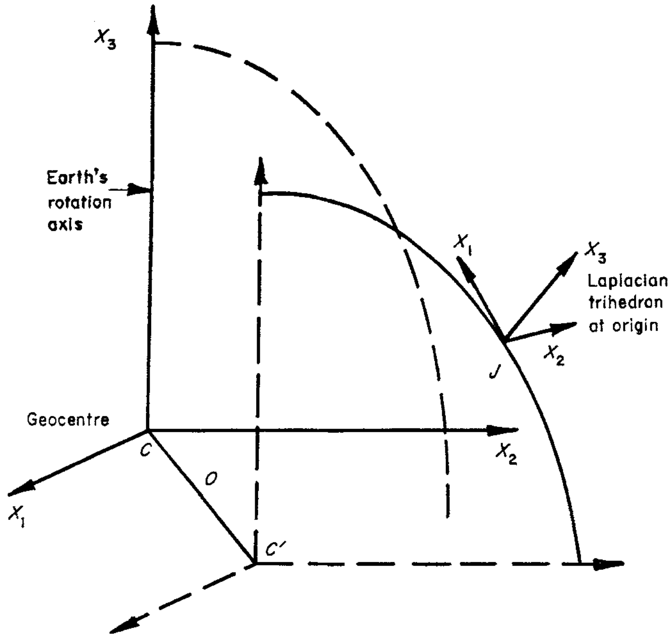

FIG. 2. The geocentric orientation vector.

ρ and ν being spheroidal radii of curvature in the meridian and prime vertical respectively. The geocentric orientation parameters $\Delta\xi_{io}$ $(i = 1, 3)$ are the changes necessary to the astrogeodetic values ξ_{iao} at the origin of the datum to obtain equivalent values on a geocentric spheroid with the same dimensions. These values ξ_{ig} are related to the former by the relation

$$\xi_{ig} = \xi_{ia} + \Delta\xi_i, \qquad i = 1, 3 \tag{24}$$

at the general point on the datum, ξ_3 being given by

$$\xi_3 = h_d. \tag{25}$$

The relation between the various components can be obtained by a study of Fig. 3 and expressed in matrix form by the equation

$$DX = A\,DX_o, \tag{26}$$

where the column matrices

$$DX = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix} \quad \text{and} \quad DX_o = \begin{pmatrix} \Delta x_{1o} \\ \Delta x_{2o} \\ \Delta x_{3o} \end{pmatrix}$$

define the components of the orientation vector in the Laplacian trihedrons at the general point P and the origin J respectively, the array A, given by

$$A = \begin{Bmatrix} \cos\phi_o\cos\phi + \sin\phi_o\sin\phi\cos\Delta\lambda & \sin\phi\sin\Delta\lambda & \sin\phi_o\cos\phi - \cos\phi_0\sin\phi\cos\Delta\lambda \\ -\sin\phi_o\sin\Delta\lambda & \cos\Delta\lambda & 0 \\ \sin\phi\cos\phi_o - \sin\phi_o\cos\phi\cos\Delta\lambda & -\cos\phi\sin\Delta\lambda & \sin\phi\sin\phi_o + \cos\phi_o\cos\phi\cos\Delta\lambda \end{Bmatrix}, \tag{27}$$

where

$$\Delta\lambda = \lambda_o - \lambda, \tag{28}$$

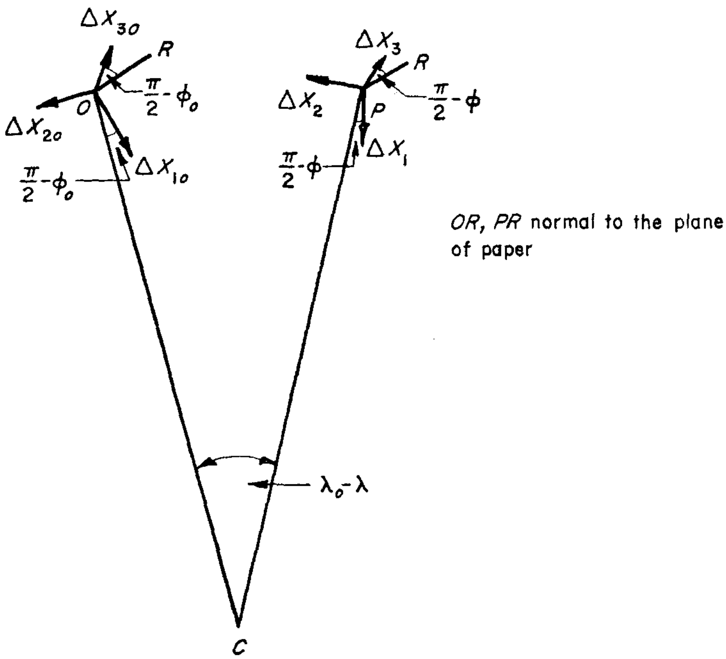

FIG. 3. Geocentric orientation parameters at origin and general point.

specifying the earth space transformation between the vector triads $\mathbf{i}_o$ and $\mathbf{i}$. The combination of equations (22), (23), (24) and (26) gives

$$\xi_{ig} = \xi_{ia} + \frac{1}{h_i} \sum_{j=1}^{3} A_{ij} h_{jo} \Delta\xi_{jo}, \qquad i = 1, 3, \tag{29}$$

A_{ij} being the appropriate element in the array A. Expanded versions of these equations are given in (Mather & Fryer 1970a, equations (8)–(11)). The quantities ξ_{ig} so obtained can be compared directly with those obtained by the use of equations (7) and (20) from gravimetric considerations. The resulting observation equations are of the form

$$v_i = \xi_{ia} - \xi_{ig} + \Delta\xi_i, \qquad i = 1, 3, \tag{30}$$

where ξ_{ig} now refer to gravimetric values. In general, three observation equations are possible at each astro-geodetic station on the datum and the resulting block of equations can be expressed in matrix form by

$$V = CX + K. \tag{31}$$

The element C_{ij} in the array C is related to the element $\{A_{rj}\}_m$ of the $(l, 3)$ array A at the m-th station in the most general case by the relation

$$C_{ij} = \frac{h_{jo}}{\{h_r\}_m} \{A_{rj}\}_m, \tag{32}$$

where

$$i = l(m-1)+r \tag{33}$$

and the element k_i in the array K is given by

$$k_i = \{\xi_{ra} - \xi_{rg}\}_m, \qquad r = 1, l; \qquad l < 3, \tag{34}$$

l being the number of parameters ξ_i being compared. Thus, if only h_d values are compared, $l = 1$; if both deflections and h_d values are compared, $l = 3$. Equation (31) is the classical block of observation equations which are solved in the usual manner for the array X, given by

$$\begin{aligned} X^T &= (\Delta\xi_{1o} \quad \Delta\xi_{2o} \quad \Delta\xi_{3o}) \\ &= (\Delta\xi_o \quad \Delta\eta_o \quad \Delta h_{do}). \end{aligned} \tag{35}$$

It has been assumed in the preceding development that ξ_{ia} and ξ_{ig} are equivalent quantities. This is correct when $i = 1$ or 2 on using equation (20). Ambiguity does occur when $i = 3$. Elevations above the spheroid are related to the results of astro-geodetic levelling through Molodenskii's equation (Molodenskii *et al.* 1962, p. 24)

$$h_{sp} = \int_{\text{geoid}}^{P} dh_o - \int_{\text{geoid}}^{P} \zeta\, dl + h_{s(\text{geoid})}, \tag{36}$$

where dh_o is the difference in orthometric elevation along the element of length dl on the route of levelling from the geoid to P, over which the mean deflection of the vertical in the direction of levelling is ζ. It can also be seen from Fig. 1 that

$$h_s = h_o + N = h_n + h_d, \tag{37}$$

where h_n is the normal height (e.g., Heiskanen & Moritz 1967, p. 171). Thus the results of astro-geodetic levelling, which comprise the second term on the right-hand side of equation (36), are not exactly equivalent to either N or h_d. This matter will be discussed further in Section 4.

3. Data sets

(i) *Surface gravity data for the Australian region* (*UNSW data set*)

The sub-divisional units specifying the data sets were the same as those adopted for the 1968 determination (Mather 1969, p. 501). The data set was supplemented by helicopter and marine gravity surveys subsequently carried out by the Bureau of Mineral Resources, Geology & Geophysics, Canberra, and information from other sources (Le Pichon & Talwani 1969; Falvey & Talwani 1969; also see acknowledgments), over 12000 additional readings resulting on the tenth degree basic grid adopted as the basis of representation. The inconsistency between the data sets described in Section 1 were eliminated as follows.

(a) Each $5° \times 5°$ square was divided into four $2\frac{1}{2}° \times 2\frac{1}{2}°$ blocks, each of which was analysed using separate two-dimensional trigonometrical series of the form (Mather 1967)

$$\Delta g_i = \sum_{j=1}^{t} C_j\{f(\phi, \lambda)\}_{ij}, \qquad i = 1, n, \tag{38}$$

where Δg_i $(i = 1, n)$ are the n available gravity values after conversion from free air anomalies to equivalent Bouguer anomalies. C_j $(j = 1, t)$ are the required coefficients.

(b) Observation equations were set up for $\frac{1}{2}° \times \frac{1}{2}°$ square means using the relation

$$\frac{1}{N} \sum_{i=1}^{N-n} \sum_{j=1}^{t} C_j\{f(\phi, \lambda)\}_{ij} = \Delta g_m + v_m - \frac{1}{N} \sum_{i=1}^{n} \Delta g_i, \tag{39}$$

n being the total number of readings available in the half degree considered, N the total number of readings possible (i.e. 25 for a basic tenth degree grid) and Δg_m is the value adopted for the square mean, given by

$$\Delta g_{mr} = \frac{1}{n_r} \sum_{i=1}^{n} \{\Delta g_i\}_r. \tag{40}$$

The value of Δg_m so computed is held fixed in the subsequent adjustment if $n > n_{\min}$ ($= 0{\cdot}4N$) and v_m is put equal to zero in equation (39) in such a case. The resulting equation is thereafter treated as a condition. The value of $0{\cdot}4N$ was adopted for $n_{\min}$ on the basis of experience in the successive stages of the compilation of the UNSW set when it was observed that reliable area means were obtained once the above degree of representation was achieved. Also see (Mather 1969, Table 4).

(c) A further condition equation was introduced to hold the predicted half degree area means to an acceptable value of the $5° \times 5°$ area mean $\overline{\Delta g}$. This equation is of the form

$$\sum_{r=1}^{t'} (\Delta g_{mr} + v_{mr}) + \sum_{i=1}^{s-t'} \Delta g_{mi} - s\,\overline{\Delta g} = 0, \tag{41}$$

where s is the number of half degree squares in a five degree square (i.e. $s = 100$) and t' is the number of half degree squares where $n < n_{\min}$. It was decided to adopt

$$\overline{\Delta g} = \frac{1}{N_t} \sum_{r=1}^{s} \sum_{i=1}^{n} \Delta g_{ir}, \tag{42}$$

where

$$N_t = \sum_{r=1}^{s} n_r \quad \text{if} \quad N_t > 1000,$$

which is the value of $n_{\min}$ for a five degree square. In cases where $N_t < 1000$, the value adopted for $\overline{\Delta g}$ was that assigned for the square in question by Rapp (1968) in his combined solution from satellite data and surface gravimetry. The details of the solution are given in (Mather 1970, pp 74–85) with some earlier material from (Mather 1967).

The end product of the above analysis was the establishment of consistent sets of $0{\cdot}1° \times 0{\cdot}1°$, $\frac{1}{2}° \times \frac{1}{2}°$, $1° \times 1°$ and $5° \times 5°$ area means for the free air anomaly field in the Australian region. An important corollary is that the values adopted to represent smaller areas less than five degrees square, are not necessarily the numerical means of all readings available within the region. In this manner, the problem of 'noise' was minimized in sparsely surveyed regions and the obvious defects in the 1968 solution were largely rectified.

(ii) *The data set used to represent the outer zone*

The outer zones were represented as in the case of the 1968 solution by the Rapp set of five degree area means which were amended as a consequence of the procedure adopted in Section 3(i). The resulting change in the zero degree term is shown in Table 1.

Table 1

Zero degree term in N_f
Reference System 1967; $W_o = U_o$; $M\{\ \}$ = Global mean value

Rapp Data set	Year	$M\{\Delta g\}$ (mgal)	$N_{fo} = -R(M\{\Delta g\}/\gamma)$ (met)
Original	1968	$+0{\cdot}5$	$-3{\cdot}2$
Amended	1970	$+0{\cdot}4$	$-2{\cdot}8$

These combined data sets can be interpreted as examples of the use of low degree harmonics of the Earth's gravity field as determined from satellite orbital analysis, for effecting field predictions using the available surface gravity values. It is also apparent, as seen in the earlier study (Mather 1969, pp 509–510), that the precision with which these low degree harmonic coefficients has a greater bearing on composite geoidal solutions than the gravity data used in the extension at the present time. The problem is therefore more akin to a spherical cap determination with surface harmonic representation of the outer zones (Molodenskii *et al.* 1962, p. 147; Cook 1950; Cook 1951) than the classical concept of a closed surface integral. The current set of computations is equivalent to the limiting radius of the spherical cap (ψ_o) being equal to approximately 20°.

Molodenskii estimated that the adoption of a surface harmonic representation up to degree 8 for the outer zone would give rise to errors not exceeding $\pm 1{\cdot}9$ m in N_f and $\pm 1{\cdot}1$ s in each of ξ_{fi} if Zhongolovich's solution were adopted for the geoid (Molodenskii *et al.* 1962, p. 162). Cook, using Jeffreys' determination of the second and third degree harmonics, estimated the residual effects of the distant zones in the case where $\psi_0 = 20°$, at $\pm 5{\cdot}1$ m in N_f (Cook 1951, p. 136) and $\pm 0{\cdot}7$ s in ξ_{fi} (Cook 1950, p. 383). de Witte (1967, pp 456–458) recommends spherical cap calculations up to $\psi_o = 39°$, which approximates to a nodal point of Stokes' function. Such an extension of the bounds of the UNSW data set was considered to be of no practical value in view of the inadequate representation available for the gravity field in the ocean areas to the south-east, south and west of Australia. Harmonic representation of the outer zones involves the use of the truncation functions of Molodenskii and Cook. This technique is well worth detailed investigation as test calculations show that computations of the separation vector using these functions up to degree 8 are a factor of 100 times faster on an electronic computer than normal surface integration techniques.

The effect of the outer zone gravity field on the final result is essentially harmonic and hence, *relative* errors in a regional study of the type undertaken will not be as large as the figures given by Molodenskii and Cook. This has been demonstrated analytically by Cook (1950, p. 374) and confirmed in the course of detailed computations in Australia (Mather & Fryer 1970a, Table 2). Harmonic coefficients whose errors will be undetected over a region like Australia are those of degree less than 6 and of orders zero and one, which have near planar variations over this area. A further possible complication is the fact that the magnitude of higher degree harmonics have not been considered when evaluating these coefficients (Cook 1965, p. 181 *et seq.*).

One method of verifying the accuracy of these low degree harmonic coefficients is by the study of the cross-covariance $M\{\Delta g_s\, \Delta g\}$ between the area means determined from satellite data alone (Δg_s) and from surface gravimetry (Δg). Tests carried out by Kaula (1966, pp 5308–5309) indicate that no serious systematic errors are likely to exist in the low degree harmonics of order zero and one which will not be detected on an Australia-wide analysis.

(iii) *Local fields around chosen astro-geodetic stations*

Thirty-eight astro-geodetic stations on the AGD were chosen to satisfy the requirements laid down in Section 1 in order that any bias in the solution due to the effect of systematic errors of prediction was minimal. The location of these stations is shown in Fig. 4. In addition it is not desirable to restrict l in equation (33) to the value 1 by comparing only separation values (ξ_3) as the values ξ_{3a} are *deduced* from astro-geodetic deflections of the vertical and, for reasons given in Section 1, are subject to interpolation errors. This is in contrast to the deflections of the vertical which are observed quantities.

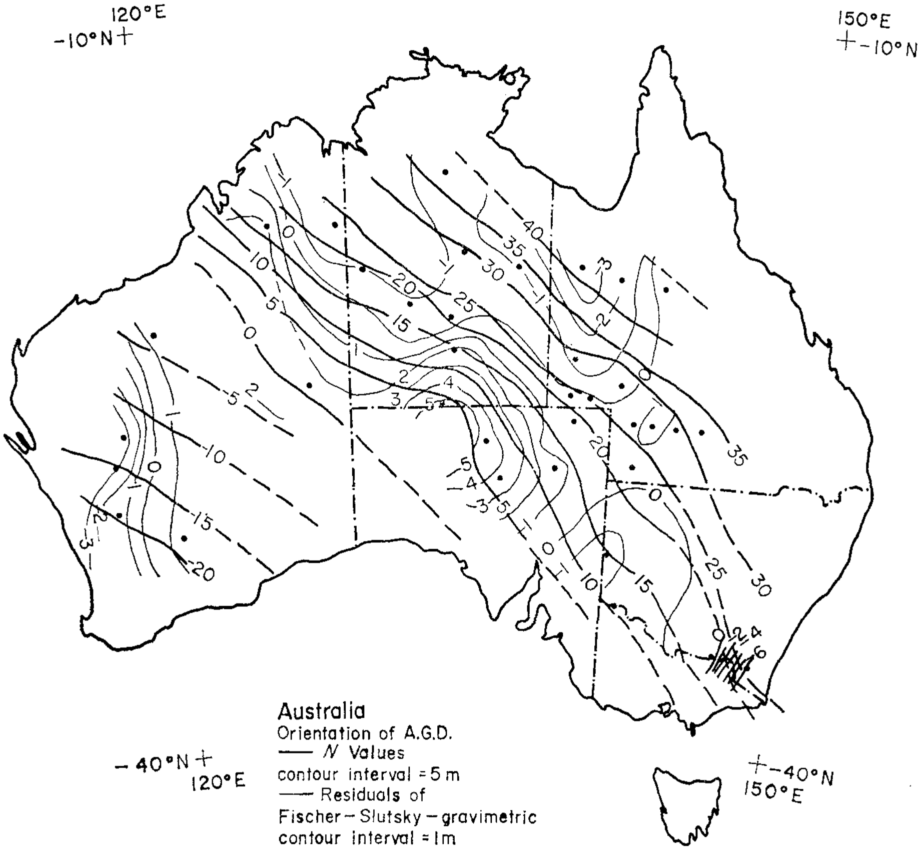

FIG. 4.

Computations of gravimetric deflections of the vertical, unlike those of separation values, are critically dependent on the accuracy with which the inner zone gravity field is represented. In the current investigation, it was decided to define the gravity field within the four tenth degree squares adjacent to the computation point with precision by using both available gravity readings as well as supplementary observations. The average spacing of gravity stations is given in Table 2, together with Shimbirev's estimate of an adequate distribution which would restrict the effect of interpolation errors on the computed value of ξ_{fi} for this region to 0·15 s (Brovar *et al.* 1964, p. 290).

Table 2

Definition of the inner zone gravity field

Distance from computation point (km)	No. of stations used Shimbirev's estimate	Average in current study
0	1	1
1·5	5	2
3·0	7	6
7·5	9	10
15	not available	18

Most astro-geodetic stations included in the present study are situated on isolated hilltops or rises on vast plains. Little error is caused in the former case by totally ignoring the existence of the hill in calculating the horizontal gravity anomaly gradients required for the evaluation of the inner zone contribution to the deflections of the vertical, the case being considered as that of a conical hill on an infinite plain. N_f values are affected as the free air anomaly is significantly correlated with elevation even in the above case. More complex topographical forms cause concern and the simple five station evaluation of the innermost zone has to be replaced by a nine-station grid from which the weighted mean of the three gradients is taken as representative of the inner zone (Rice 1952, p. 290).

These ideal requirements were not always complied with due to access difficulties and lack of time on field expeditions. The resulting loss of accuracy has a negligible effect on the final result. The inner zone gravity field was digitized in the form of a uniform $0{\cdot}01° \times 0{\cdot}01°$ grid established by prediction from a Bouguer anomaly representation of the available gravity stations using the technique set out in (Mather 1967). The region adjacent to the computation point whose effect was evaluated by the use of horizontal gravity anomaly gradients, was represented by the nine-hundredth degree squares symmetrical with respect to the square containing the astro-geodetic station (Mather 1969, p. 504). The contributions of the other 391 hundredth degree squares comprising the rest of the zone were computed using equations (8) and (14). Consistency of the UNSW data set was maintained by replacing the tenth degree square values by the mean of the relevant hundredth degree values.

4. The results

The principal contributor to ξ_{ig} in equation (30) is the free air geoid, while ξ_{ia} $(i = 1, 2)$ are the astro-geodetic deflections of the vertical on the AGD established under the direction of the Division of National Mapping. The astro-geodetic determination of Fischer & Slutsky (1967) is used to represent ξ_{3a} in the current investigation. The accuracy of ξ_{1a} and ξ_{2a} are almost totally dependent on the precision of the astronomical determinations in the case of the AGD as the geoid and spheroid are almost parallel and lack coincidence by only about ten metres. The rms errors of the procedures adopted by the Division of National Mapping have previously been estimated at $\pm 0{\cdot}6$ s in ξ_{1a} and $\pm 1{\cdot}0$ s in ξ_{2a} (Mather & Fryer 1970a, Section 4). The precision of the Fischer–Slutsky solution is discussed in (Mather 1969, p. 514).

It can be deduced from the development in Section 2 that any non-Stokesian effects are predominantly local in character. The use of the maximum coverage of the datum ensures that the exclusion of topographical terms in the solution will not significantly affect the geocentric orientation parameters. Further the weaknesses in the elevations of gravity stations masks the distinctions between normal and orthometric elevations. It can therefore be safely assumed that the results of astro-geodetic levelling are directly comparable with gravimetric values represented by the free air geoid N_f over 90 per cent of the Australian region without taking into account the differences which exist in their precise definition, the free air geoid being well known to be a good approximation to both N and h_d (Mather 1968a).

Two methods of solution were used.

(i) *Comparisons at* 38 *pre-selected stations*

The free air geoid, computed by the use of equations (8) and (14) was used as the gravimetric solution (ξ_{ig}) in establishing the observation equations defined by equation (24) at (38) selected astro-geodetic stations mentioned in Section 3(iii). The resulting geocentric orientation parameters, defined in equation (35), were used in equation (29) to compute the geocentric equivalents of the astro-geodetic quantities. The residuals which resulted are shown in Figs 4–6. The rms residuals for different types of solution are shown in Table 3, being defined by equation (3).

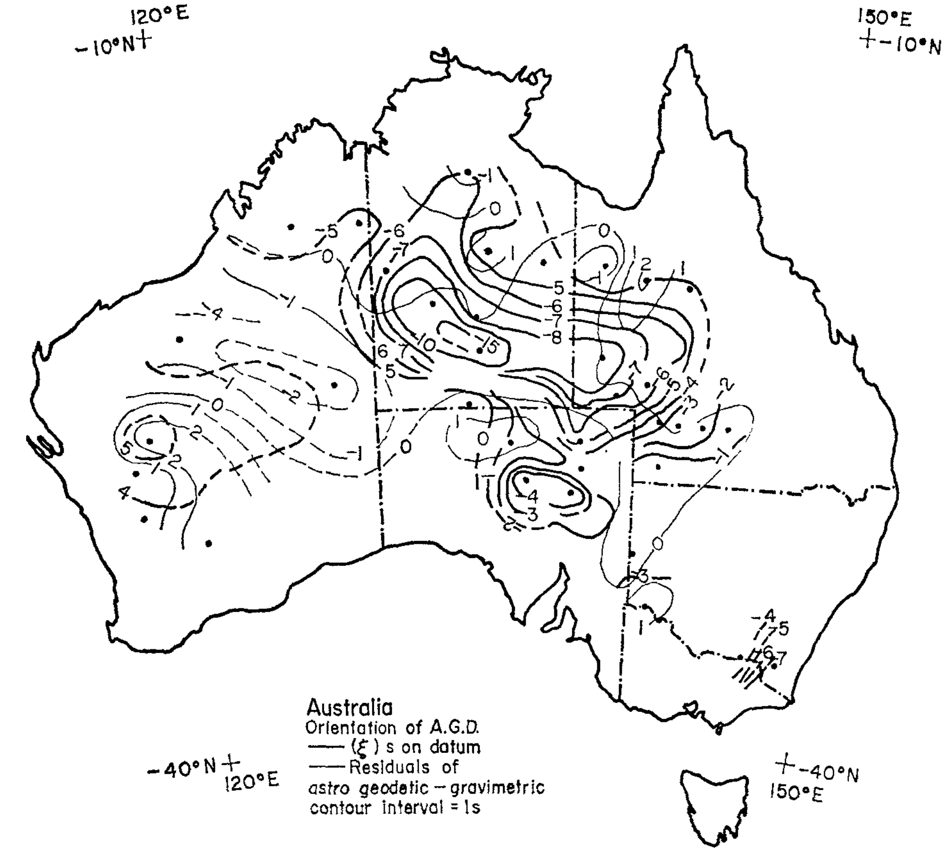

FIG. 5.

Four different types of solution are used to obtain estimates of the geocentric orientation parameters ($\Delta\xi_{io}$, $i = 1, 3$) and called *types* 1–4 in Table 3. Only solutions of types 1–3 apply to evaluations at the 38 selected stations. Type 4 is described in Section 4(ii). In type 1 solutions only separation (ξ_3) values were used in setting up the observation equations, while in type 2, only deflections of the vertical (ξ_1, ξ_2) were considered. Type 3 solutions involved all three parameters in determining $\Delta\xi_{io}$.

Three *classes* of weight coefficients w_i were used for the observation equations in ξ_i ($i = 1, 3$). Those in class A solutions comprise the uniform set

$$w_1 = 1{\cdot}5; \qquad w_2 = 1{\cdot}0; \qquad w_3 = 3{\cdot}0, \tag{37}$$

which is based on the conclusions drawn from a previous study (Mather & Fryer 1970a, Section 4). The set of coefficients given by

$$w_i^{-1} = \sigma_i^2 = M\{(\xi_{ia} + \Delta\xi_i - \xi_{ig})^2\}, \qquad i = 1, 3, \tag{38}$$

were used in class B solutions. The mean square residuals used in equation (38) were based on a class A solution but of the same type. For example, the weight coefficients used in Table 3, code 4, which is type 1, class B had the numerical values

$$w_1 = 0{\cdot}75; \qquad w_2 = 0{\cdot}28; \qquad w_3 = 0{\cdot}20.$$

It is interesting to note that comparatively large changes in uniform weight coefficients have negligible effects on the values of $\Delta\xi_{io}$, as can be seen from an

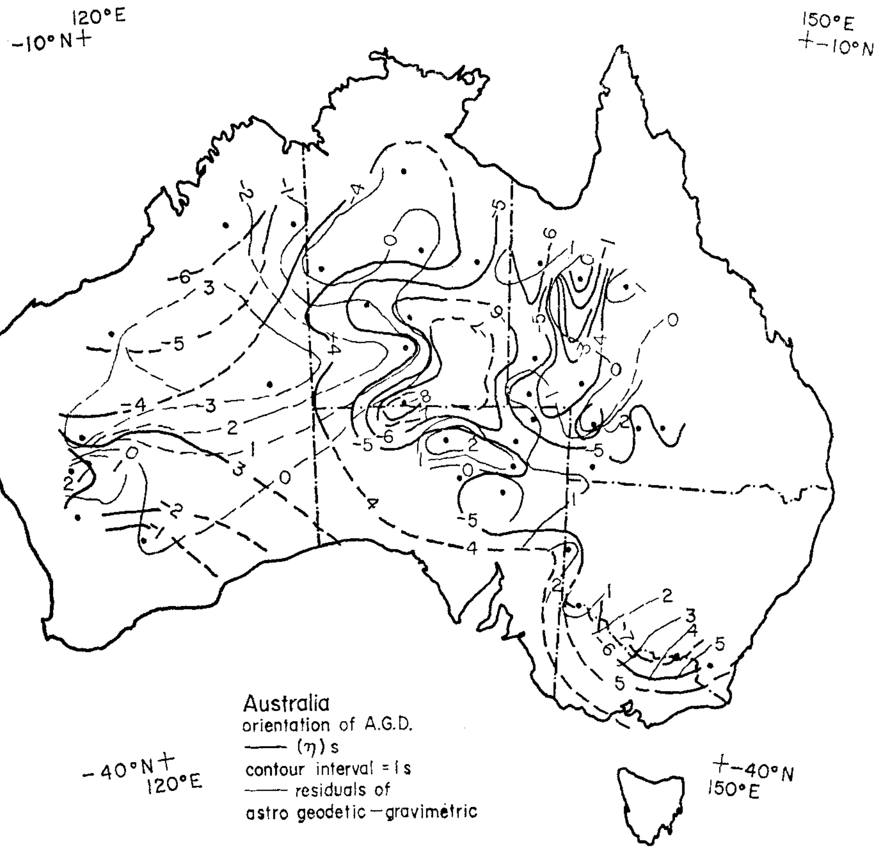

FIG. 6.

examination of Table 3, codes 3 and 4. The values of w_i in class C solutions were computed by the use of the relation

$$w_i^{-1} = (\xi_{ia} + \Delta\xi_i - \xi_{ig} - \sigma_i)^2, \quad i = 1, 3, \tag{39}$$

the values on the right-hand side of equation (39) being based on a class B solution of the same type. In this case, each observation equation has a different value for w_i.

The results obtained from class C solutions do not give the smallest residuals as those points at which comparatively large discrepancies occur tend to be weighted out of the solution. As these mainly occur on the peripheries of the datum due to weak gravity fields at sea, the solution is consequently biased towards the comparisons at the centre of the continent. Such an orientation is not desirable, as discussed in Section 1, due to the existence of systematic error in predicted values of the gravity field.

The nine solutions discussed so far provide very similar values of the orientation parameters $\Delta\xi_{io}$. The apparent stability of the solution was further tested in two ways. In the first, the contributions of all regions within $1\frac{1}{2}°$ of the computation point were excluded from the gravimetric solution, the resulting determination being listed in Table 3, codes 19 and 20. Two interesting observations were made from this solution.

(a) The orientation parameters obtained were only marginally greater than those from a full solution (e.g., Table 3, code 9). It could therefore be concluded that the computation of non-Stokesian terms in the separation vector are not necessary

Table 3

Solutions for the geocentric orientation parameters

Code	Description	Type	Class	$\Delta\xi_{1o}$	ξ_1(s) σ_1	$\Delta\xi_{2o}$	ξ_2(s) σ_2	$\Delta\xi_{3o}$	ξ_3(m) σ_3
1	Planar approximation	1	A	0·0	−0·5±3·9	−0·5	−0·7±2·6	−3·5	0·0±2·7
2	Planar approximation	2	A	0·5	0·0±3·8	0·0	0·0±2·6	—	—
3	38 stations	1	A	−4·0	0·3±1·0	−4·3	0·4±1·8	9·8	0·0±2·2
4	38 stations	1	B	−4·1	0·2±1·0	−4·3	0·4±1·8	9·8	−0·0±2·3
5	38 stations	1	C	−3·9	0·4±1·0	−4·3	0·5±1·8	10·7	0·8±2·2
6	38 stations	2	A	−4·2	0·0±1·2	−4·5	0·0±1·6	—	—
7	38 stations	2	B	−4·2	0·0±1·2	−4·5	0·0±1·6	—	—
8	38 stations	2	C	−3·7	0·5±1·2	−4·4	0·0±1·5	—	—
9	38 stations	3	A	−4·0	0·3±1·0	−4·3	0·4±1·8	9·8	0·0±2·2
10	38 stations	3	B	−4·1	0·2±1·0	−4·3	0·4±1·8	9·8	0·0±2·3
11	38 stations	3	C	−3·9	0·4±1·0	−4·3	0·5±1·8	10·7	0·8±2·2
12	693 points on a 1° grid	4	—	−4·2	—	−4·4	—	10·1	0·0±2·5
13	Code 12 at 38 stns	3	—	−4·2	0·2±1·0	−4·4	0·3±1·8	10·1	0·5±2·3
14	Composite-codes 3 & 6	3	—	−4·2	0·1±1·1	−4·5	0·3±1·8	9·8	0·2±2·4
15	Composite	3	—	−4·3	0·0±1·1	−4·7	0·1±1·8	9·8	0·4±2·6
16	Code 9 on 1° grid	4	—	−4·0	—	−4·3	—	9·8	−0·4±2·7
17	Code 14 on 1° grid	4	—	−4·2	—	−4·5	—	9·8	−0·3±2·5
18	Code 15 on 1° grid	4	—	−4·3	—	−4·6	—	9·8	−0·2±2·7
19	Excl. inner cap $\Psi > 1\frac{1}{2}°$	1	A	−3·8	0·4±3·3	−4·2	0·2±2·5	10·0	0·0±1·6
20	Excl. inner cap $\Psi > 1\frac{1}{2}°$	2	A	−4·2	0·0±3·4	−4·2	0·0±2·4	—	—
21	Excl. outer zone $\Psi < 20°$	1	A	−3·3	−0·4±1·5	−3·7	−0·6±2·3	1·3	0·0±2·8
22	Excl. outer zone $\Psi < 20°$	2	A	−2·7	0·0±1·3	−2·8	0·0±1·7	—	—

for the determination of the geocentric orientation parameters, to the order of the flattening, as they can be treated as purely local errors. This conclusion must, however, be accepted with reservation especially if there is a systematic variation in the nature of the topography across the region being investigated.

(b) The rms residual of comparisons with the astro-geodetic geoid was significantly smaller than usual, while those for the deflections of the vertical increased in magnitude. This indicates that the astro-geodetic solution is more smoothened with respect to the geoid than the gravimetric one.

A similar procedure was carried out with a gravimetric solution restricted to a spherical cap of limiting radius 20° and the results are shown in Table 3, codes 21 and 22. The exclusion of the outer zone is shown to give a weaker fit in all of ξ_1, ξ_2 and ξ_3. A comparison with the determination at code 3 shows that the rms residual ξ_3 increases from $\pm 2{\cdot}25$ to $\pm 2{\cdot}83$ m, the outer zone contributing about $\pm 1{\cdot}7$ m through the rms residual towards improving the match between the gravimetric and astro-geodetic solutions.

No definite conclusions can be drawn from the above tests until some assessment is made of the current location of the AGD in earth space. This may be obtained by effecting the solution defined by equations (24)–(35) with all gravimetric values held constant, for example, at the value zero. The resulting solutions, shown at Table 3, codes 1 and 2, give the orientation parameters necessary to correct the datum to the mean geoid slope across it *as represented by the* 38 *stations included in the study*. From the results of code 1 it can be seen that a correction of $-3{\cdot}45$ m to the astro-geodetic geoid height at the Johnston origin, presently held at zero, together with corrections $0{\cdot}00$ s to ξ_{1_o} and $-0{\cdot}50$ to ξ_{2_o} would transform the AGD to lie in the mean geoid slope across the datum. These figures are in close agreement with values estimated by Fischer and Slutsky for the last two terms (1967, p. 331). The code 2 solution which is independent of the Fischer–Slutsky determination, indicates that the corrections to ξ_{1_o} and ξ_{2_o} should be $+0{\cdot}48$ s and $-0{\cdot}01$ s respectively. These figures, which could have been predicted from the mean residuals in the previous solution, are not equivalent to Bomford's mean deflections of the vertical (Bomford 1967, p. 57–58), which are straight numerical means, while the former are values after local vector triads have been corrected to the Laplacian triad at the origin.

It can therefore be concluded that:

(i) The adequate representation of the outer zone gravity field is critical in the determination of the geocentric orientation parameters; and

(ii) It would suffice if only reasonable representation of the inner zone gravity fields were available around selected stations in obtaining these parameters with adequate precision.

It is evident that the free air geoid alone will provide the required parameters $\Delta\xi_{i_o}$ with a precision approaching $\pm 0{\cdot}2$ s which cannot be exceeded in view of the limitations inherent in the representation presently available for the gravity field. The non-Stokesian terms due to the topographic effects would be included in the residuals obtained by the use of equation (30). The study of such residuals could well become an integral part of physical geodesy as they are a consequence of the comparisons between the results obtained from *two completely independent methods* and afford models of little ambiguity for the study of a variety of problems, provided the astronomical observations are of adequate precision.

(ii) *Comparisons on a one degree grid*

A study of Table 3 shows that the orientation parameters obtained for ξ_1 and ξ_2 on comparing deflections alone (type 2 solutions) are slightly different from the results

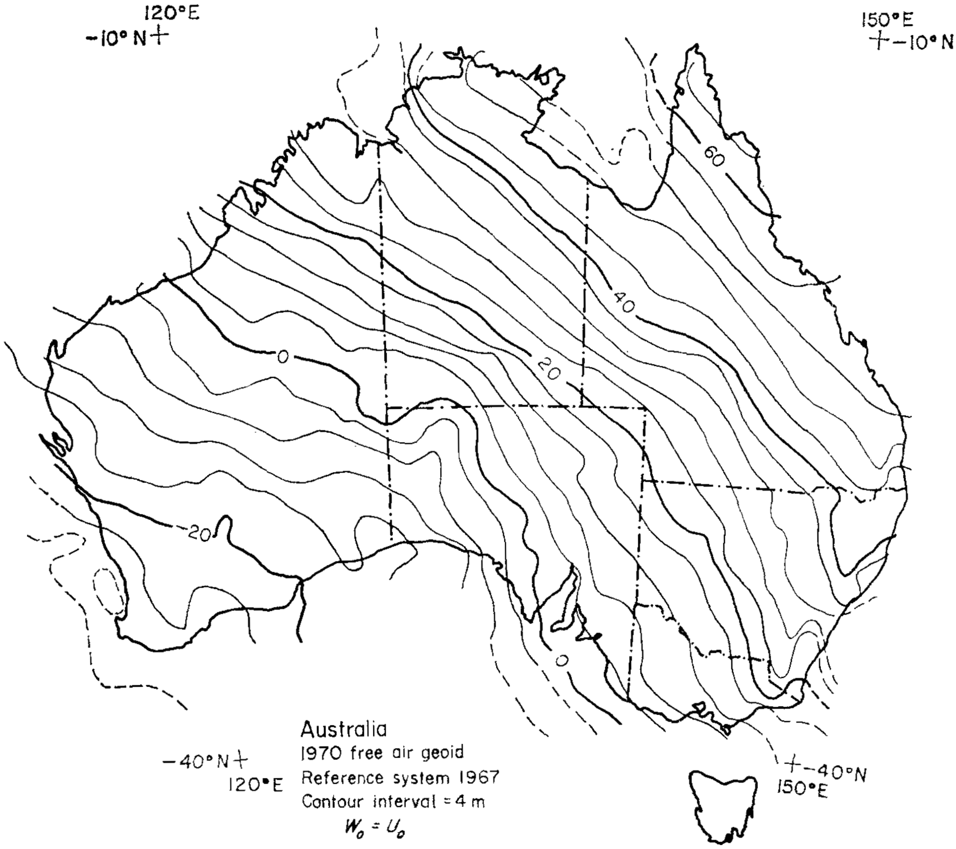

FIG. 7.

of types 1 and 3 which include the astro-geodetic geoid of Fischer & Slutsky. The values at code 1, which are *independent of the gravimetric solution*, indicate that the geoid in vicinity of the Johnston origin is depressed with respect to the mean geoid slope on the basis of the directions of the vertical at the 38 stations included in the study.

This figure is also borne out by solutions, classified as type 4 in Table 3, obtained by the comparison of ξ_3 values only between the two solutions at the corners of a one degree grid at 693 points evenly spaced over the Australian mainland, as N_{f_0} at the Johnston origin is 5·7 m. The gravimetric determination used in the type 4 solutions is called the 1970 *free air geoid for Australia* and shown in Fig. 7, the resulting orientation parameters being listed in Table 3, code 12. The residuals in ξ_3 are shown in Fig. 8 which is equivalent to Fig. 10 in (Mather 1969). A comparative assessment of these figures shows that the major inconsistencies which existed in comparisons using the 1968 solution have been elimінted by the adoption of the technique outlined in Section 3 for effecting the necessary predictions in preparing the revised data set. The rms residual ξ_3 for the 1970 solution is $\pm 2{\cdot}5$ m over the entire continent (Table 3, code 12) in comparison to $\pm 5{\cdot}2$ m obtained from the 1968 solution. It is also evident from Fig. 8 that systematic discrepancies still exist between the two solutions but over limited extents and, with one exception, on the peripheries.

The exception is the geoidal low over the Officer Basin in South Australia which shows up clearly on both the astro-geodetic determination of Fischer & Slutsky as well as on the 1970 free air geoid after conversion to the AGD, as shown in Fig. 9.

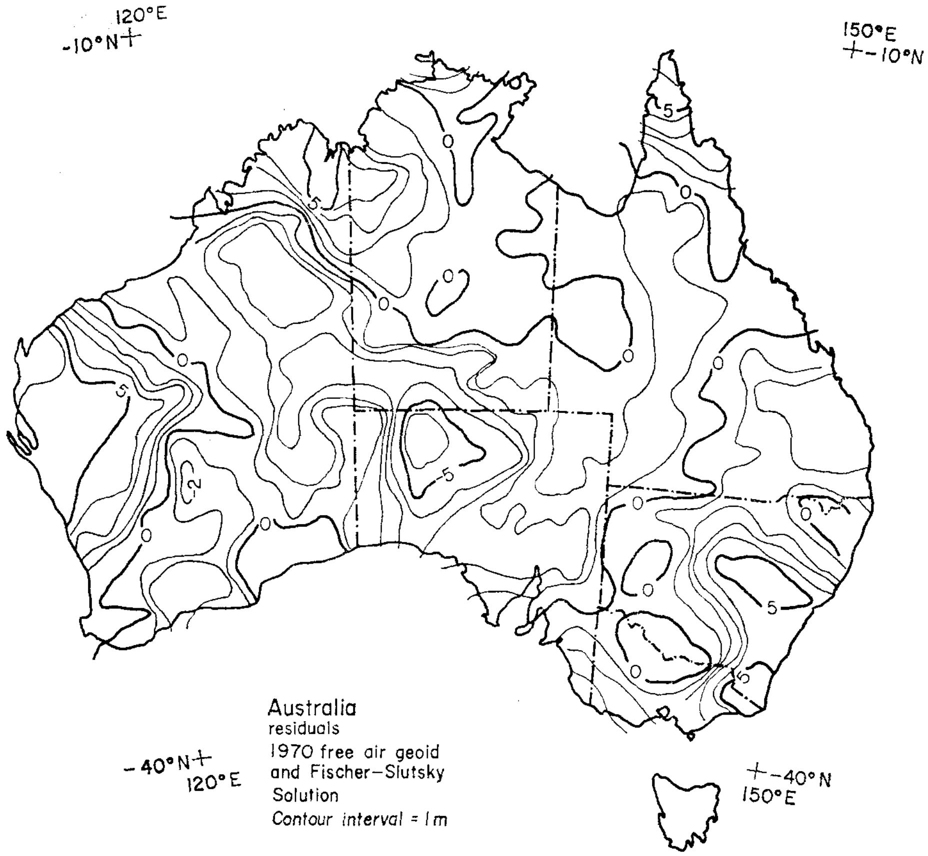

FIG. 8.

However, the minimum as obtained from gravimetry is approximately 7 m in excess of the value obtained from astro-geodesy. Another point of interest is the greater geoidal high obtained from gravimetry in the Snowy Mountains region at the south-west. It is apparent that the solution of Fischer & Slutsky is of commendable accuracy when assessed in terms of the density of one station per 10000 km^2 and the uneven distribution. Also see Section 4(i)(b). There is also little doubt that the accuracy of the gravimetric determination is such that the drawing of any conclusions regarding the sources of the residuals remains a matter for speculation.

(iii) *The best set of orientation parameters*

A study of Table 3 shows that, on exclusion of type 3 solutions which are disproportionately representative of the datum as a whole for reasons given in Section 4(i), values can be assigned for the geocentric orientation parameters with confidence so that they lie within a range of 20 cm in ξ_3 (i.e. separation) and 0·2 s in each of ξ_1 and ξ_2. The exact values to be adopted still require clarification.

As earlier tests indicated that inner zone contributions had marginal effects on the orientation parameters $\Delta\xi_{i_0}$, it was decided to include type 4 solutions in selecting the best values for $\Delta\xi_{i_0}$. It can also be seen that the values of $\Delta\xi_{1_0}$ and $\Delta\xi_{2_0}$ obtained from such solutions are not significantly different from those obtained in type 2 solutions (code 6). In addition solutions of types 1 and 3, which include the astro-geodetic geoid, are identical for a given class of weight coefficient. It was therefore

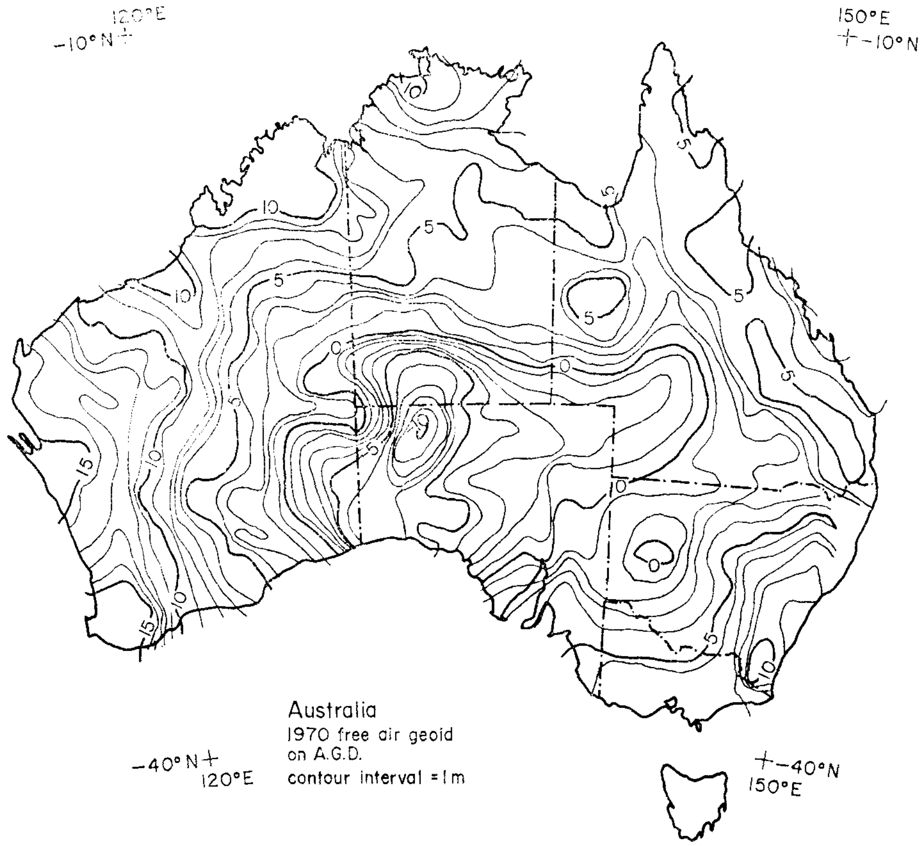

FIG. 9.

concluded that the astro-geodetic geoid at the 38 stations was not representative of the datum as a whole and a number of composite solutions were investigated with *minimal concern for slight increases in the rms residual in* ξ_3 *at the* 38 *stations.*

Two composite solutions were considered. The first was defined by the geocentric orientation parameters

$$O = \{\Delta\xi_{1o} = -4{\cdot}22\,\mathrm{s};\ \Delta\xi_{2o} = -4{\cdot}54\,\mathrm{s};\ \Delta\xi_{3o} = 9{\cdot}82\,\mathrm{m}\}$$

being a combination of codes 3 and 6, the resulting rms residuals being given at code 14. These results are only marginally different to those at code 13 where the values of $\Delta\xi_{io}$ were adopted from 12 which was based on a type 4 solution. The forced mean values of the residuals in ξ_1 and ξ_2 are a measure of the non-representative nature of the residuals in ξ_3. The incorporation of these forced means into the orientation parameters results in comparisons whose rms residuals are given in code 15. Changes occur in the rms residuals of ξ_3 but not in those of ξ_1 and ξ_2 as envisaged.

It can therefore be concluded that the following set of geocentric orientation parameters best fit the Australian Geodetic Datum as represented by the astro-geodetic data available

$$\mathbf{O} = \{\Delta\xi_{1o} = -4{\cdot}2\pm0{\cdot}2\,\mathrm{s};\ \Delta\xi_{2o} = -4{\cdot}5\pm0{\cdot}2\,\mathrm{s};\ \Delta\xi_{3o} = 10{\cdot}0\pm0{\cdot}2\,\mathrm{m}\}. \qquad (40)$$

This set differs significantly from that obtained by the use of the 1968 free air geoid (Mather 1969, p. 515). The earlier results should be disregarded in view of the flaws known to exist in the 1968 data set.

The error estimates given in equation (40) are based on detectable errors and do not reflect the magnitude of any systematic errors in low degree harmonic coefficients of orders zero and one as discussed in Section 3(ii). Even degree zonal harmonics must be considered the most reliable of the suspect terms having been evaluated from secular variations. Accuracy estimates based on comparisons between Rapp's solution for these coefficients from surface gravimetry alone and the Smithsonian Astrophysical Institution Standard Earth solution (Rapp 1969b, p. 231) indicate that the maximum error possible in $\Delta\xi_{3_o}$ is ± 5 m while those in $\Delta\xi_{1_o}$ and $\Delta\xi_{2_o}$ are $\pm 0{\cdot}4$ s. These limiting values would be smaller if the error estimates were closer to values given by Cook for those in C_{20} and C_{40} (Cook 1965, p. 181).

The accuracy of the value adopted for $\Delta\xi_{3_o}$ is also affected by zero degree implications arising from the possibility that $W_o \neq U_o$. U_o which enters into the solution through equation (8), is based on the parameters defining RS 1967. These include, in addition to the spheroid of reference, values adopted for the constant kM and the angular velocity of rotation of the Earth. The former is given by

$$kM = 3{\cdot}986\,03 \times 10^{20}\,\mathrm{cm}^3\,\mathrm{s}^{-2}$$

for RS 1967. Comparisons with other recent determinations (e.g., Rapp 1967) indicate an uncertainty of 3 parts in 10^6 in this value. The influence of the term

$$\frac{W_o - U_o}{\gamma}$$

in equation (8) as critical as it contributes linearly to $\Delta\xi_{3_o}$, being dependent on (a) the accuracy of the value adopted for kM in defining the reference system; and (b) whether the spheroid of reference has the same volume as the geoid.

The above term can be evaluated if the assumption (b) is valid (Mather 1968a, p. 528), when its magnitude is not expected to exceed 5 m. The validity of this assumption appears to be questionable by a factor of three which is linearly equivalent to the uncertainties in $\Delta\xi_{1_o}$ and $\Delta\xi_{2_o}$. It is well known that the gravimetric method by itself cannot define the separation vector with an accuracy in excess of the values given in equation (40), with the proviso that the fundamental uncertainty is a zero degree effect on a global scale and therefore constant for all datums. The incorporation of the geocentric orientation vector for each of the major datums into a global adjustment using geometrical satellite triangulation techniques would produce a non-dynamic evaluation of the term $(W_o - U_o)/\gamma$ and hence provide a value for the potential W_o of the geoid. Such a procedure would remove the uncertainties attendant in interpreting the results of conventional satellite triangulation due to the higher degree terms in the surface harmonic representation of the geoid/spheroid separation having a tendency to be correlated with the distribution of topography. For other remarks on the subject see (Molodenskii *et al.* 1962, pp. 78 *et seq*; Heiskanen & Moritz 1967, pp. 100 *et seq*).

The final set of geocentric orientation parameters defining the geocentric orientation vector **O** at the Johnston origin of the AGD, on inclusion of the zero degree term from Table 1 are

$$\mathbf{O} = \begin{pmatrix} \Delta\xi_{1_o} = \Delta\xi_o & = -4{\cdot}2 \pm 0{\cdot}2\,\mathrm{s} \\ \Delta\xi_{2_o} = \Delta\eta_o & = -4{\cdot}5 \pm 0{\cdot}2\,\mathrm{s} \\ \Delta\xi_{3_o} = \Delta N_o = & 7{\cdot}2 \pm 0{\cdot}2\,\mathrm{m} \end{pmatrix}, \tag{41}$$

the estimate of the precision of $\Delta\xi_{3_o}$ not taking into account zero degree effects or any gravitational effects which have near planar variations over the Australian region.

5. Conclusions

(i) *The* 1970 *free air geoid for Australia*

The geoid across Australia can be related to a geocentric spheroid if the set of free air gravity anomalies used in the computation has no harmonics of first degree and first-order of second degree on global surface harmonic analysis. The zero degree term in such a determination requires a knowledge of the potential W_0 of the geoid. The accuracy with which W_0 can be computed depends entirely on that with which the volume of the geoid is known. If such zero degree considerations are neglected, the height anomaly, the geoid-spheroid separation and the results of astro-geodetic levelling can be assumed to be equivalent for the Australian region on allowing for any transformation of datum that is necessary, as the systematic errors which are inevitable in the representation of the incompletely surveyed global gravity field, provide sufficient 'noise' to drown out all indirect effect considerations.

The resulting solution, called the 1970 free air geoid for Australia is based on a composite data set where the gravity field within 20° of the Australian coast has been defined either by observations or predictions in accordance with the criteria developed in Section 3. Such a solution based on compatible data sets provides a good approximation to both the height anomaly and the geoid/spheroid separation. Tests of the 1970 free air geoid with the astro-geodetic determination of Fischer and Slutsky indicated that the former was a marked improvement over the 1968 solution, the rms residual of comparisons of the separation reducing from $\pm 5{\cdot}3$ m to $\pm 2{\cdot}5$ m. Tests indicate that the accuracy of the gravimetric solution is at least on par with that of an astro-geodetic determination prepared from an average station spacing of 100 km.

(ii) *The set of geocentric orientation parameters for the AGD*

The composite data set described in Section 3 was used in two ways to obtain the required parameters. In the first case, the latter were obtained by comparing gravimetric values of the deflections of the vertical ξ_1 and ξ_2 as well as the separation ξ_3 at 38 astro-geodetic stations evenly spaced over the datum and shown in Fig. 4, with astro-geodetic values. The procedure was also repeated on a one degree Australia-wide grid using only ξ_3 values, the AGD being represented by the solution of Fischer and Slutsky. The geocentric orientation parameters so obtained defined the vector $C'C$ in Fig. 2. The values obtained by the second method were in good agreement with a composite set established by the first, this procedure being resorted to as the astro-geodetic geoid deduced at the 38 stations was not representative of the entire datum.

The following conclusions were drawn after testing various possible sets of orientation parameters.

(a) The inner zone field has only a marginal effect on the values determined for a region the size of Australia.

(b) The non-Stokesian topography dependent terms have purely local effects to within the precision sought in the present investigation though care should be taken when there is a distinct correlation between topography and position over large extents.

(c) The use of the free air geoid alone as the total gravimetric solution should give orientation parameters with errors less than $\pm 0{\cdot}1$ s in $\Delta\xi_{1_0}$ and $\Delta\xi_{2_0}$ and $\pm 0{\cdot}1$ m in $\Delta\xi_{3_0}$ provided the gravity field is adequately defined with no serious systematic errors either in the predicted field or in the representation adopted for the low degree harmonics of the gravity field. The current data set cannot be said to meet these requirements as can be seen from equation (41).

(d) The rms residual of the comparisons between the free air geoid and the astro-geodetic solution after datum translation varies between $\pm 2{\cdot}2$ m at the 38 stations to $\pm 2{\cdot}5$ m over the continental extent. The significance of these figures must be evaluated in the light of the following observations.

(i) The astro-geodetic geoid matches a gravimetric solution excluding inner zone effects *with smaller residuals* than a complete solution, indicating that the former is over-smoothened as a result of the low station density of 1 per 10000 km^2.

(ii) The residuals are position dependent as shown in Fig. 8, inferring the existence of systematic error in both the astro-geodetic and the gravimetric solutions. The errors in the former, due to over-smoothing, result in the under-estimation of geoidal highs with a reverse effect for lows, as can be seen in the Officer Basin region near the Johnston origin. Those in the gravimetric solution are due to correlated prediction errors which continue to be of significance in determinations at certain regions on the continental margin.

(e) The best set of geocentric orientation parameters which can be deduced from the 1970 data set is given by equation (41) which also includes the effect of the zero degree term but assumes that the potential of the geoid is equal to that of the reference spheroid.

(iii) *Conclusion*

It should be noted that the bulk of the UNSW data set was compiled from Bouguer anomaly maps and the free air anomalies were generated using the digital representation of topographical maps available in 1964. Such a procedure was warranted in view of the gaps existing the gravity anomaly field presently available and confirmed by the above results. The only restrictions on the gravimetric determination of the geocentric orientation vector are the available gravity field and the extent to which prediction has to be resorted to. The systematic component in the error of prediction is proportional to the interval from the nearest observed value and hence results which are based on extensive predictions within ten degrees of the computation point are significantly affected. The magnitude of the effect for a given element of surface area decreases with increase of distance from the computation point till the first nodal point of Stokes' function.

The chances of obtaining satisfactory predictions from purely statistical processes alone are limited. It is preferable to adopt a combination of statistical and analytical techniques for this purpose, the latter assessing the regional trends while the former estimates the likely deviations of individual values from the trend. From a geodetic point of view it would be most desirable if marine gravity surveys were to establish 'noise' free estimates of area means at discrete intervals in order that analytical techniques could be successfully applied during the interim period preceding the complete global representation of the Earth's gravity field.

In the short term, gravimetric determinations of the geocentric orientation vector will provide both a valuable independent check on any global satellite triangulation scheme and a means of improving the estimate of the potential of the geoid. In addition, the method provides a means for the definition of local geodetic datums with respect to acceptable invariants in earth space with attendant long-term geophysical benefits.

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